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On the Numerical Determination
of Shrinkage Stresses

Frances Bauer and Edward L. Reiss

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ON THE NUMERICAL DETERMINATION OF SHRINKAGE STRESSES

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Abstract

An iterative method is employed to obtain numerical solutions of a two dimensional mixed boundary value problem for rectangular elastic plates. The theory of plane stress is used. This problem is encountered when the edge of a uniformly heated plate is bonded to a rigid body and then allowed to cool. The numerical results are compared with two formal asymptotic expansions of the solution. They correspond to the limiting cases of a thin strip and a long strip. The development of boundary layers near the edges of the plate is observed. The numerically determined stresses achieve their largest values at the corner of the bonded edge.

1. Introduction.

A thin rectangular elastic plate is heated uniformly and then an edge is bonded to a rigid half-space. In this paper we present a numerical method to determine the shrinkage stresses which develop when the plate cools.

We assume that the deformations of the plate are adequately described by the theory of plane stress. Thus we must solve a mixed boundary value problem, which we call Problem S, for the two dimensional displacement equations of equilibrium. It is solved by a previously developed [1] accelerated iteration method.* Each iterate is the solution of a mixed boundary value problem for Poisson's equation. The iterates are numerically determined by approximating this boundary value problem by difference equations. The resulting system of linear algebraic equations is of quasi-tridiagonal form. It is solved by a direct factorization method [4]. The procedure is suitable for use with a digital computer. It is applicable to other shrinkage problems with more complicated geometries.

The method is applied to rectangular plates with aspect ratios, $1/8$, $1/4$, $1/2$, 1 . Without loss of generality we assume that the plate lies in the x,y plane and that the bonded edge coincides with the line $x = 0$, $|y| \leq a$. Results of the computations are discussed in Section 4 and they are

* Also see references in [1] for previous applications of the method. Other applications are given in [2,3].

shown in Figures 1-3. They are compared with formal asymptotic expansions of the solution of Problem S as $a \rightarrow \infty$ and as $a \rightarrow 0$. As $a \rightarrow \infty$ a boundary layer occurs at $y = \pm a$ and as $a \rightarrow 0$ a boundary layer occurs at $x = 0$. The formal expansions are derived in the Appendix.

The stresses on the boundary are discontinuous at the corners $x = 0, y = \pm a$. The numerically determined stresses achieve their largest values at and near this point. Elsewhere in the plate the normal stress in the y direction is usually the largest. An approximate solution of Problem S has been previously obtained [5] by truncating a formal infinite series representation of the solution. The results are in qualitative agreement with those presented in this paper except near $x = 0, y = \pm a$. At this point the series predicts unbounded stresses. The precise behavior of the solution of Problem S at $x = 0, y = \pm a$ is not known.

2. Formulation.

The mid-plane of the plate occupies the rectangle: $0 < x < 1, |y| < a$ in the x, y plane. The edge $x = 0, |y| \leq a$ is bonded to the rigid half-space $x \leq 0$. The remainder of the boundary is stress free. Without loss of generality, we assume that the coefficient of thermal expansion of the plate is 1. Since the plate is uniformly heated the solution is symmetric with

respect to the line $y = 0$. Therefore we need only consider the rectangle R : $0 < x < 1$, $0 < y < a$. We denote the boundary of R by B .

The two-dimensional displacement vector $\underline{u}(x,y)$ has the components $u(x,y)$ and $v(x,y)$. The vector displacement equation of equilibrium for plane stress is [6]:

$$\Delta \underline{u} = -c \nabla (\nabla \cdot \underline{u}) , \quad c \equiv (1+\nu)/(1-\nu) . \quad (1)$$

and Hooke's law is

$$\begin{aligned} \sigma_x &= E^* (u_x + \nu v_y), \quad \sigma_y = E^* (v_y + \nu u_x), \quad \sigma_{xy} = (E^*/2)(1-\nu)(u_y + v_x), \\ E^* &\equiv E/(1-\nu^2) . \end{aligned} \quad (2)$$

Here E and ν are Young's modulus and Poisson's ratio and $\sigma_x(x,y)$, $\sigma_y(x,y)$ and $\sigma_{xy}(x,y)$ are the stresses. Subscripts on u and v denote partial derivatives and subscripts on σ denote components of the stress.

Since the plate is bonded to a rigid body, we have the boundary conditions

$$u = 0 , \quad v = y , \quad \text{on } x = 0 , \quad 0 \leq y \leq a . \quad (3a)$$

The stress free conditions yield

$$u_x + \nu v_y = u_y + \nu u_x = 0 , \quad \text{on } x = 1 , \quad 0 \leq y \leq a , \quad (3b)$$

$$v_y + \nu u_x = u_y + \nu u_x = 0 , \quad \text{on } 0 \leq x \leq 1 , \quad y = a . \quad (3c)$$

The symmetry conditions are

$$u_y = v = 0, \quad \text{on } 0 \leq x \leq 1, \quad y = 0. \quad (4)$$

The shrinkage boundary value problem,

Problem S, is to determine two functions $u(x,y)$ and $v(x,y)$ with continuous second partial derivatives in R that satisfy the differential equations (1) in R and the boundary conditions (3).

3. The Numerical Method.

We seek an approximate solution of Problem S by an accelerated iteration method [1]. Thus starting from an initial estimate $u^0(x,y)$ of the solution we define a sequence of iterates u^n by the recursions

$$\begin{aligned} \Delta \bar{u}^n &= -c \nabla(\nabla \cdot \bar{u}^{n-1}), & \text{for } x,y \text{ in } R, \\ \bar{u}^n &= 0, \quad \bar{v}^n = y, & \text{on } x = 0, \quad 0 \leq y \leq a, \\ \bar{u}_x^n &= -v_y^{n-1}, \quad \bar{v}_x^n = -u_y^{n-1}, & \text{on } x = 1, \quad 0 \leq y \leq a, \\ \bar{v}_y^n &= -v u_x^{n-1}, \quad \bar{u}_y^n = -v_x^{n-1}, & \text{on } 0 \leq x \leq 1, \quad y = a, \\ \bar{u}_y^n &= \bar{v}^n = 0, & \text{on } 0 \leq x \leq 1, \quad y = 0, \\ \bar{u}^n &= \theta \bar{u}^n + (1-\theta) \bar{u}^{n-1}, & \text{for } x,y \text{ in } R+B. \end{aligned} \quad (5)$$

Here θ is the acceleration parameter. When $\theta = 1$ we have simple iterations. We find that for other values of θ the

iterations converge more rapidly. The vector $\tilde{u}^n$ is a provisional iterate.

Each iterate in (5) is determined by solving the following mixed boundary value problems for Poisson's equation,

$$\begin{aligned} \Delta w &= f(x,y) , & \text{for } x,y \text{ in } R, \\ w &= g(y) , & \text{on } x = 0, 0 \leq y \leq a, \\ w_x &= h(y) , & \text{on } x = 1, 0 \leq y \leq a, \\ w_y &= p(x) , & \text{on } 0 \leq x \leq 1, y = a , \end{aligned} \tag{6}$$

and on $0 \leq x \leq 1, y = 0$,

$$\text{either } w_y = 0 , \quad \text{or } w = 0 .$$

Here f , g , h and p are evaluated from the data and from previous iterates.

We solve (6) by a finite difference method. Thus R is covered by a uniform rectilinear mesh with mesh width δ in the x and y directions and the mesh lines are parallel to the edges. The mesh is selected so that the edges of the rectangle coincide with mesh lines. The mesh points x_i, y_j are defined by

$$x_i = i\delta , \quad i = 0,1,\dots,I, \quad y_j = j\delta, \quad j = 0,1,\dots,J, \tag{7}$$

where $I = a/\delta$, $J = b/\delta$. The set of mesh points in R is called R_δ and the set of mesh points on B is called B_δ . At each point of $R_\delta + B_\delta$ the solution of (6), $w(x_i, y_j)$, is approximated by the mesh function w_{ij} . It is the

solution of a system of linear algebraic equations obtained by approximating (6) by appropriate difference equations. Thus, the Laplacian in (6) is approximated in R by the 9-point Laplace difference operator [7] and the boundary normal derivatives in (6) are approximated by three point inward difference formulae. The derivatives that are required to evaluate f, g, h and p are approximated by centered difference formulae.

The coefficient matrix of the algebraic equations for w_{ij} is tridiagonal with respect to matrices. The system is solved by a direct factoring method [4]. Some of the details of the method are described elsewhere [1].

When the iterations have converged for a fixed value of a, the stresses are computed at each mesh point from finite difference approximations of (2). A numerical estimate of an appropriate value of δ was determined from a sequence of test calculations with successively smaller values of δ .

The following numerical convergence criterion was employed,

$$\max_{(x_i, y_j) \in R_\delta + B_\delta} |u_{ij}^n - u_{ij}^{n-1}| < 10^{-8}. \quad (8)$$

In a test calculation the right side of (8) was decreased. Only insignificant changes in the answers occurred. The condition (8) is only necessary for convergence.

The number of iterations that are required for convergence varies with a, δ and the choice of θ . We select

θ so as to minimize the number of iterations. For most of the calculations we used $\theta = .6$. For $a = 2$ with $\delta = 1/20$, 797 iterations were required to satisfy (8). This used approximately 1.5 minutes of CDC 6600 computer time.

4. Presentation and Discussion of the Results.

The iteration method was applied to rectangles with $a = 4, 2, 1, 1/2$. The mesh width $\delta = 1/20$ was employed for $a = 4, 2, 1$, and $\delta = 1/40$ was used for $a = 1/2$. Results of the computations^{*} are presented in Figures 1-3. Here we use the dimensionless stresses, $S_x \equiv \sigma_x/E^*$, $S_y \equiv \sigma_y/E^*$ and $S_{xy} \equiv 2\sigma_{xy}/[E^*(1-\nu)]$. In these figures the thin strip and long strip approximations that are derived in the Appendix are denoted by $a = \infty$ and $a = 0$. The thin strip approximation is a formal asymptotic expansion of the solution of Problem S as $a \rightarrow \infty$.^{**} It can only be valid if y is bounded away from a . Near $y = a$ there is a boundary layer in which

* The computations were performed on the CDC 6600 Computer at the AEC Computing and Applied Mathematics Center of the Courant Institute of Mathematical Sciences. In all calculations we used $\nu = .32$.

** We observe that the plane stress theory may give an inaccurate approximation of the three dimensional stresses as $a \rightarrow \infty$. In [8] it was shown that the plane stress theory gives the leading term away from the edges in a formal asymptotic expansion of the solution of the three dimensional theory of elasticity as the thickness h of the plate approaches zero, and the other dimensions of the plate are fixed. If the present problem is considered as three-dimensional, there is a double limit: $h \rightarrow 0$ and $a \rightarrow \infty$.

the solution of Problem S varies rapidly from the thin strip approximation to satisfy the boundary conditions at $y = a$. The long strip approximation is a formal asymptotic expansion as $a \rightarrow 0$. It can only be valid if x is bounded away from zero. Near $x = 0$ there is a boundary layer.

The variation with y of the stresses at the bonded edge $x = 0$ is shown in Figures 1. We observe that they approach the thin strip approximation

$$s_x^t = s_{xy}^t = 0, \quad s_y^t = 1 - \nu^2 = .8976, \quad (9)$$

as a increases if y is bounded away from a . The stresses vary rapidly from (9) as $y \rightarrow a$. Since S_y and S_{xy} vanish on $y = a$ (see 3c) there is a discontinuity in the stresses on the boundary at $x = 0, y = a$. The rapid variation near $y = a$ is further illustrated in Figure 2 where the stresses on the line $x = 1/2$ are shown for $a = 4$.

S_y is the largest stress over most of the interior of the plate. The largest numerically determined stresses are the values of S_x near the corner $x = 0, y = a$. The stresses determined by the formal infinite series representation [5] are unbounded at this point. Of course, the stresses that are determined by the present numerical method are bounded. For the rectangle with $a = 2$, numerical solutions were obtained with the mesh widths $\delta = 1/20$, $\delta = 1/26$ and $\delta = 1/40$. The stresses were in close agreement except near $x = 0, y = a$.

The stresses at this point are shown in Table I.

Table I

δ	1/20	1/26	$\delta = 1/40$
S_x	4.914	5.462	6.434
S_y	2.470	2.645	2.956
$- S_{xy}$	4.856	5.207	5.823

Thus the stresses at $x = 0$, $y = a$ increase substantially as δ decreases.

In Figures 3 we show the variation with x of the stresses on the symmetry line $y = 0$. As a increases the stresses approach (9). As a decreases the stresses approach the long strip approximation near $x = 1$. Even for $a = 1/2$ the formation of a boundary layer near $x = 0$ can be observed. Presumably, the boundary layer will be more pronounced for smaller values of a .

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Appendix. The Thin Strip and the Long Strip Approximations.

We define new variables η and $\mathcal{U}(x, \eta)$ by

$$\eta = y/a, \quad \mathcal{U}(x, \eta) \equiv a^{-1}u(x, y), \quad (10)$$

where the vector $\mathcal{U}$ has components U and V . We assume that the solution $\mathcal{U}$ of Problem S has the asymptotic expansion

$$\mathcal{U} \sim \sum_{j=0}^{\infty} \mathcal{U}^j(x, \eta) a^{-j}, \quad (11)$$

as $a \rightarrow \infty$. The expansion coefficients $\mathcal{U}^j$, $j = 0, 1, \dots$, are determined by inserting (10) and (11) into (1) and (3) and equating to zero the coefficients of each power of a . Thus we find that U^j and V^j satisfy the following recursive system of equations:

$$U_{xx}^j = - (1+c)^{-1} (V_{x\eta}^{j-1} + U_{\eta\eta}^{j-2}), \quad (12)$$

$$V_{xx}^j = - c U_{x\eta}^{j-1} - (1+c) V_{\eta\eta}^{j-2}, \quad (13)$$

$$U^0 = 0, \quad V^0 = \eta, \quad U^j = V^j = 0, \quad j > 0, \quad \text{on } x = 0, \quad 0 \leq \eta \leq 1, \quad (14)$$

$$U_x^j = -v V_{\eta}^{j-1}, \quad V_x^j = -U_{\eta}^{j-1}, \quad \text{on } x = 1, \quad 0 \leq \eta \leq 1, \quad (15)$$

$$v U_x^j = -V_{\eta}^{j-1}, \quad V_x^j = -U_{\eta}^{j-1}, \quad \text{on } 0 \leq x \leq 1, \quad \eta = 1, \quad (16)$$

$$U_{\eta}^j = V^j = 0, \quad \text{on } 0 \leq x \leq 1, \quad \eta = 0, \quad (17)$$

where we define $\mathcal{U}^j \equiv 0$ if $j < 0$.

Setting $j = 0$ in (12) and (13) we find that U^0 and V^0 are linear functions of x with coefficients that are functions

of η . These coefficients are evaluated from (14) and (15) with $j = 0$. Thus we obtain

$$U^0 \equiv 0, \quad V^0 = \eta. \quad (18)$$

The boundary condition (17) with $j = 0$ is satisfied by (18). We insert (18) in (12-15) with $j = 1$ and obtain

$$U^1 = -v_x, \quad V^1 \equiv 0. \quad (19)$$

Equation (17) with $j = 1$ is satisfied by (19). By induction using (12-15) and (18) and (19) it is easy to show that

$$U^j = V^j \equiv 0, \quad j \geq 2. \quad (20)$$

The thin strip approximation,

$$u^t = -v_x, \quad v^t = y, \quad (21a)$$

which we denote by superscript t , is obtained by inserting (18-20) in (11) and using (10). The stresses of the thin strip approximation are obtained from (2), (10) and (21a). They are given by

$$s_x^t = s_{xy}^t = 0, \quad s_y^t = 1 - v^2. \quad (21b)$$

The thin strip approximation does not satisfy the first boundary condition in (3c) at $y = a$. Hence (21) is not a uniform asymptotic expansion of the solution of Problem S as $a \rightarrow \infty$. It can only be asymptotic to the solution if y is bounded away from a . Near $y = a$ the solution of Problem S must

vary rapidly from (21) as $a \rightarrow \infty$ so as to satisfy the boundary conditions at $y = a$. Thus there is a boundary layer at the edge $y = a$ as $a \rightarrow \infty$. An analysis of the boundary layer shows that the stresses in the boundary layer are determined by solving plane stress problems for semi-infinite strips. These problems are as difficult to solve as the original Problem S. Thus we shall not consider the boundary layer approximation. The rapid variation near $y = a$ of the solution of Problem S is seen in Figures 1 and 2.

The long strip approximation is obtained by substituting the change of variables (10) in Problem S and assuming that the solution has the asymptotic expansion

$$\underline{u} \sim \sum_{j=0}^{\infty} \underline{v}^j(x, \eta) a^j, \quad (22)$$

as $a \rightarrow 0$. The $\underline{v}^j$ must satisfy a recursive system of equations. By analyzing this system in the same way as we did (12-17) we find that

$$\underline{v}^j = 0, \quad j \geq 0. \quad (23)$$

Thus the long strip approximation is $\underline{u}^{\ell} \sim 0$. It does not satisfy the second boundary condition (3a) at $x = 0$. Hence it can be asymptotic to the solution of Problem S only for x bounded away from zero. Near $x = 0$ there is a boundary layer. We shall not analyze the boundary layer. The development of the boundary layer can be seen from the numerical

results presented in Figure 3. The physical interpretation of the long strip approximation is that for small a , the boundary constraint at $x = 0$ influences the stresses only near $x = 0$. Thus most of the plate is essentially free to contract as it is cooled and no stresses are developed.

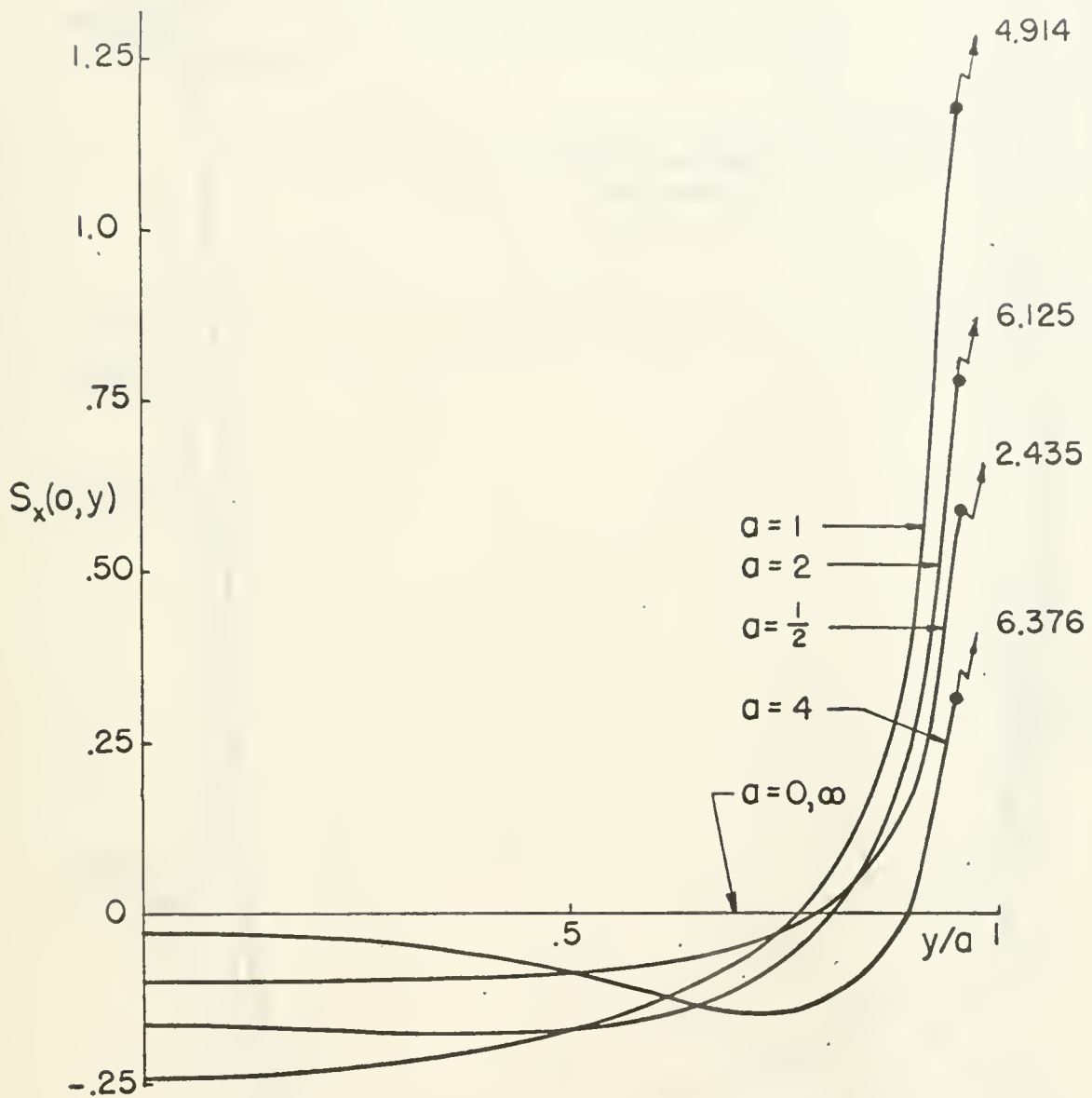
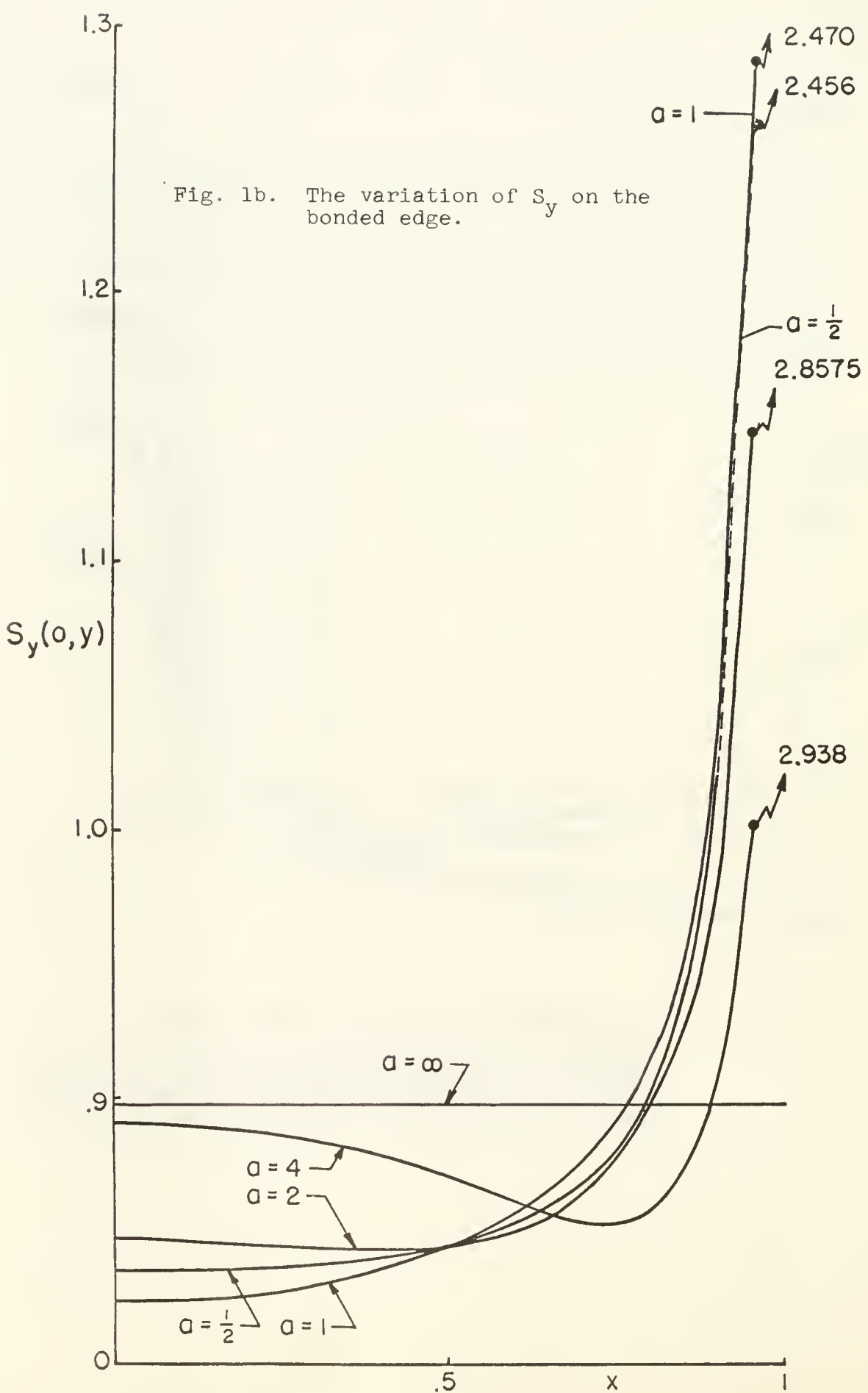


Fig. 1a. The variation of S_x on the bonded edge $x = 0$.
 The thin strip approximation is denoted by $a = \infty$ and the long strip approximation by $a = 0$. The numbers adjacent to the vertical arrows are the numerically determined stresses at the edge $y = a$.



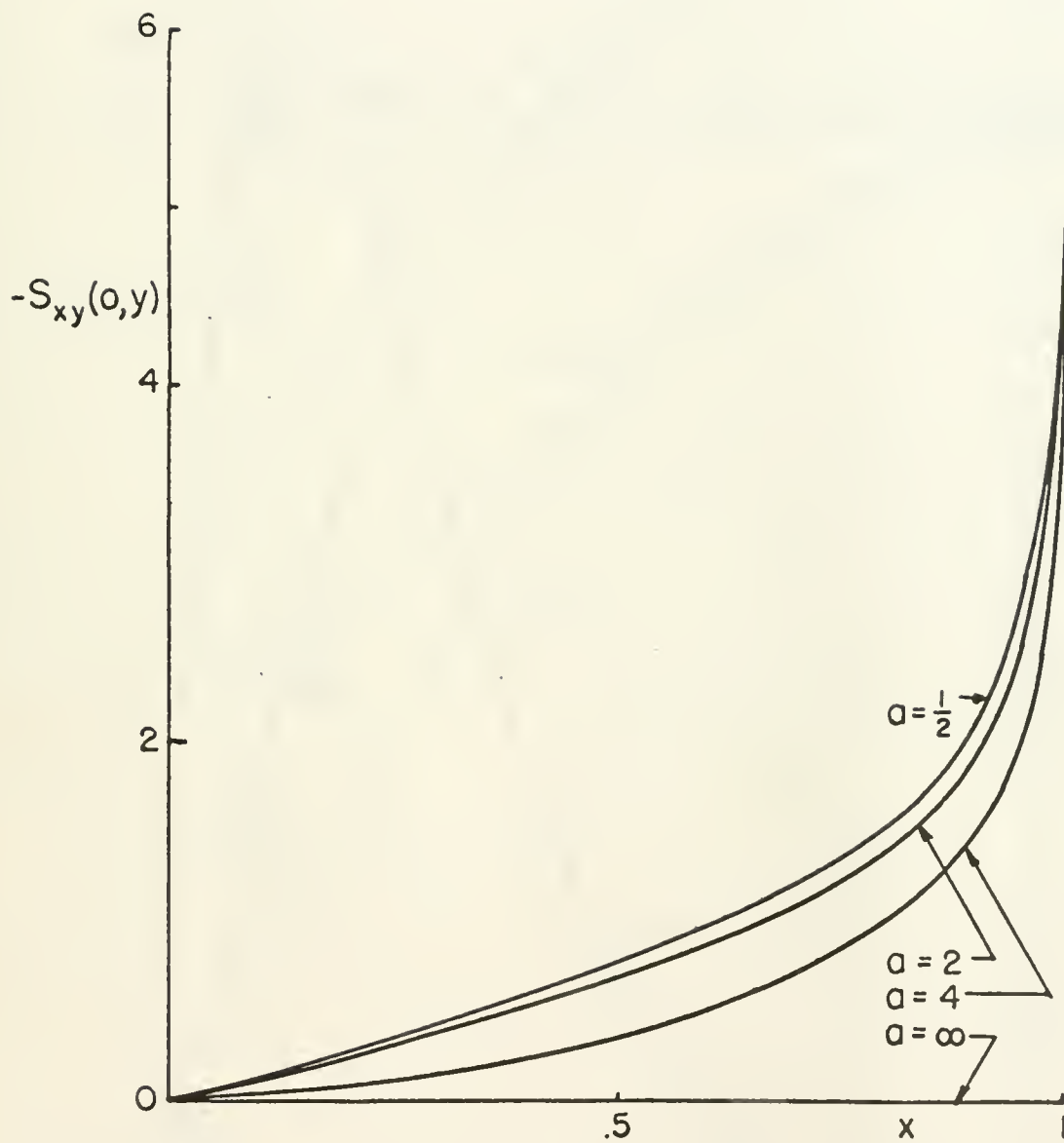


Fig. 1c. The variation of S_{xy} on the bonded edge.

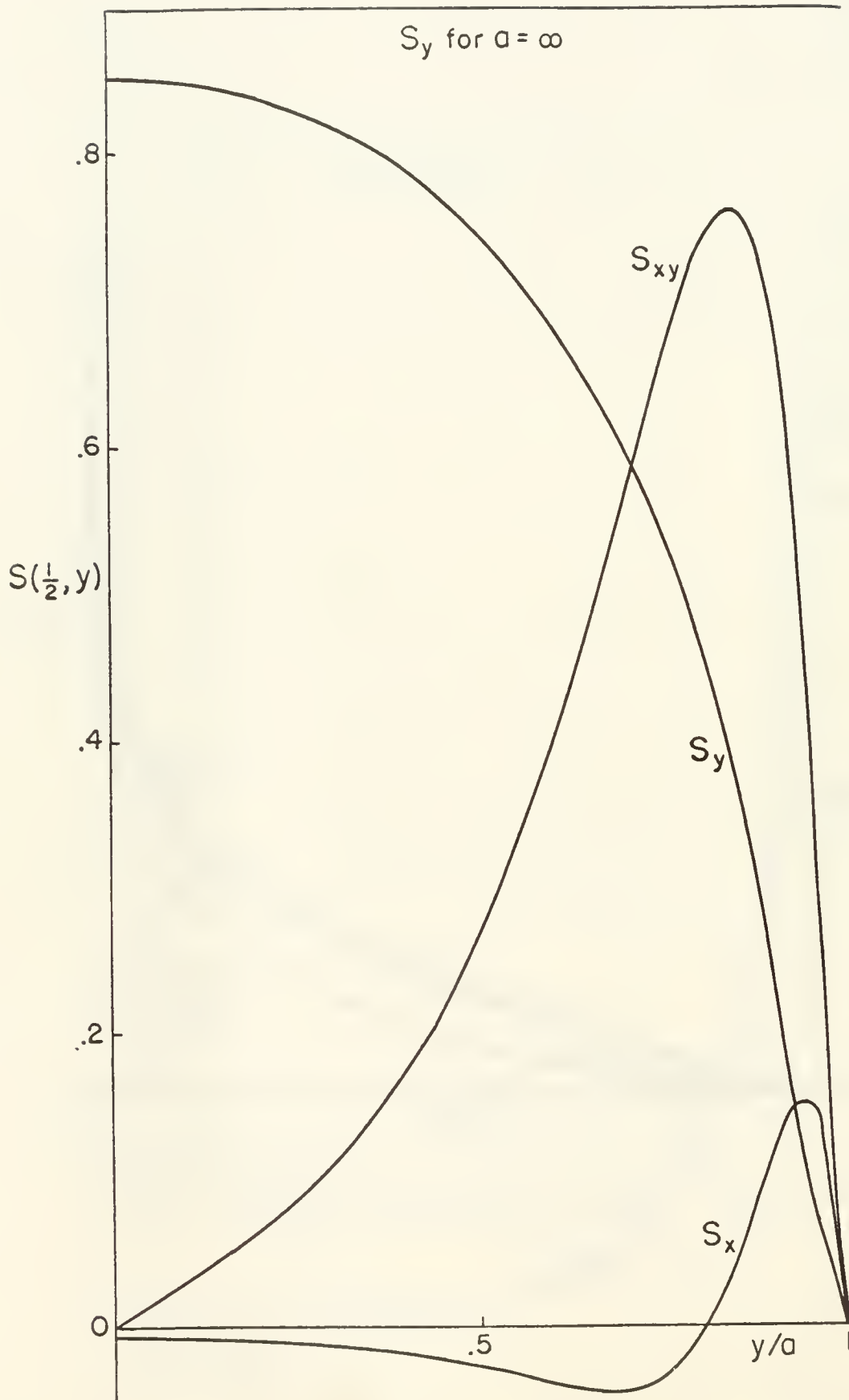


Fig. 2. The variation of the stresses along the width of the plate at $x = 1/2$ for $a = 4$.

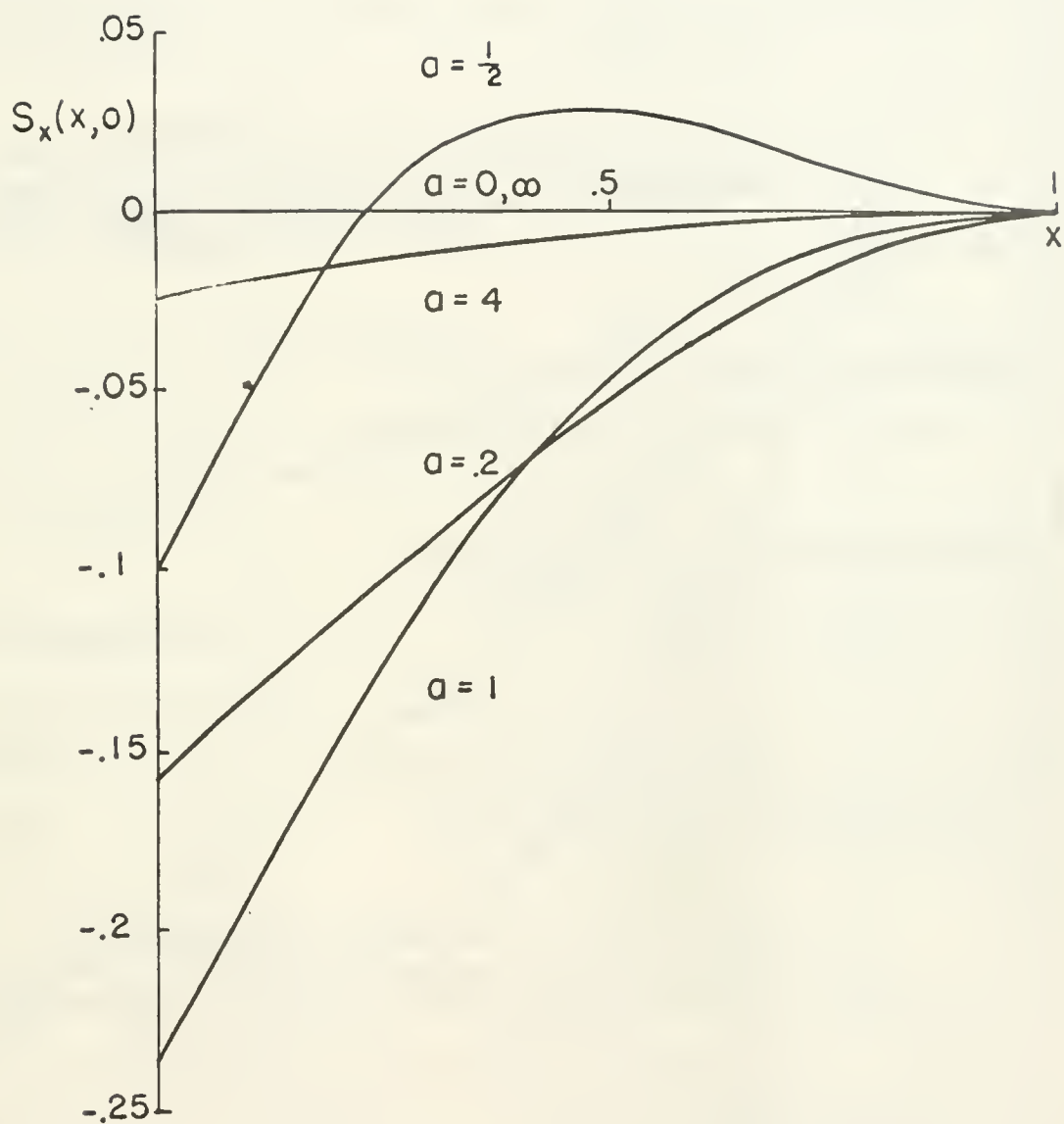


Fig. 3a. The variation of S_x along the length of the plate at the line of symmetry $y = 0$.

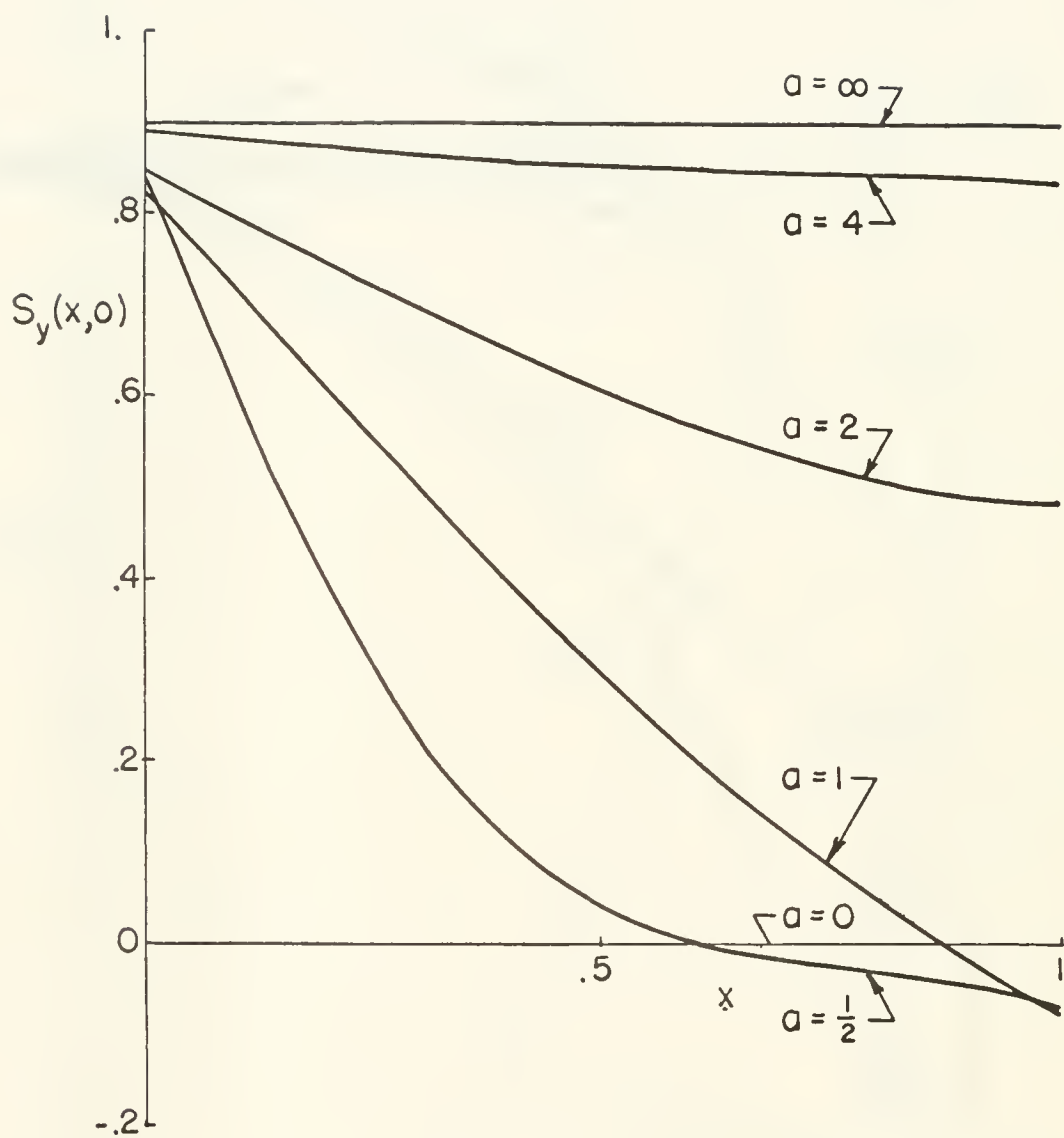


Fig. 3b. The variation of $S_y(x,0)$.

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